A NOTE ON CONVEX CONFORMAL MAPPINGS

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Abstract. We establish a new characterization for a conformal mapping of the unit disk $\mathbb{D}$ to be convex, and identify the mappings onto a half-plane or a parallel strip as extremals. We also show that, with these exceptions, the level sets of λ of the Poincaré metric $\lambda|dw|$ of a convex domain are strictly convex.

The purpose of this short article is to present a new sharp characterization of conformal mappings of the unit disk $\mathbb{D}$ onto convex domains with some implications for the Poincaré metric of the image. In particular, we will improve on an inequality obtained in [3], where the classical characterization of convexity

$$\operatorname{Re}\left\{1+z\frac{f''}{f'}(z)\right\}\geq 0 \tag{1}$$

was shown to imply the stronger inequality

$$\operatorname{Re}\left\{1+z\frac{f''}{f'}(z)\right\}\geq \frac{1}{4}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2 . \tag{2}$$

Let Sf be the Schwarzian derivative of f. Our first result is that, in fact,

Theorem 1. *The function f is a convex mapping of $\mathbb{D}$ if and only if*

$$\operatorname{Re}\left\{1+z\frac{f''}{f'}(z)\right\}\geq \frac{1}{4}(1-|z|^2)\left(2|Sf(z)|+\left|\frac{f''}{f'}(z)\right|^2\right) . \tag{3}$$

If equality holds at a single point in (3) then it holds everywhere and f is a mapping either onto a half-plane or a parallel strip.

Note that (3) reduces to (2) when f is a Möbius transformation.

A second result is an application of (3) to a property of the Poincaré metric for convex regions. Recall that the Poincaré metric on $f(\mathbb{D})$ is defined by $\lambda(w)|dw| = |dz|/(1-|z|^2)$, $w = f(z)$. We will show that, except for a half-plane or a parallel strip, the level sets of λ have strictly positive curvature, or equivalently that the sets in $\mathbb{D}$ defined by $(1-|z|^2)|f'(z)| =$ constant have this property relative to the conformal metric $|f'||dz|$. The presence of the Schwarzian term in (3) is crucial for establishing this.

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Theorem 2. *The level sets of $\lambda(w)$ in a convex domain have nonnegative curvature. If the curvature of any level set is zero at a single point then the domain is a parallel strip or a half-plane and all level sets have zero curvature.*

This refines the results in [1] and [5], where it is shown that on convex regions the function $1/\lambda$ is concave, or equivalently that $\log\lambda$ is convex. It follows from these earlier results that the sets $\lambda \le c$ are convex, but it does not rule out flat parts of the curve $\lambda = c$ or isolated points where the curvature vanishes.

Proof of Theorem 1. The sufficiency follows at once as (3) is stronger than (1). Suppose next that f is convex. Via (1) we know that

$$1 + z\frac{f''}{f'}(z) = \frac{1+h(z)}{1-h(z)}$$

for some holomorphic $h\colon \mathbb{D} \to \mathbb{D}$ with $h(0) = 0$. We appeal to Schwarz's lemma. The function $\varphi(z) = h(z)/z$ is holomorphic, maps $\mathbb{D}$ into $\overline{\mathbb{D}}$, and

$$\varphi(z) = \frac{f''(z)/f'(z)}{2 + zf''(z)/f'(z)}\,.$$

One possibility is $|\varphi| \equiv 1$. In this case f is a half-plane mapping, $Sf = 0$, and (3), really (2), holds with equality for all z.

If $|\varphi| \not\equiv 1$ then

$$\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \le \frac{1}{1-|z|^2}. \tag{4}$$

This implies

$$(1-|z|^2)^2|Sf(z)| + 2\,|p(z)|^2 \le 2 \tag{5}$$

after a short calculation, where we have written

$$p(z) = \bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z).$$

In turn, on expanding $|p(z)|^2$, (5) can be rearranged to yield (3) (and vice versa). The two inequalities are equivalent, but the important point for our work is that in (3) the factor $1-|z|^2$ occurs to the first power, not to the second.

Suppose now that equality holds in (3) at one point, and suppose also that $|\varphi| < 1$ (the case $|\varphi| \equiv 1$ having been analyzed). Equality in (3) at a point implies equality in (5) at a point, and then also equality in (4) at a point. Thus φ is a Möbius transformation of $\mathbb{D}$ to itself and equality holds everywhere in (3), (4) and (5). Furthermore, it follows from Lemma 1 in [3] that f maps $\mathbb{D}$ onto a parallel strip.

□

The proof shows that (5) is also a necessary and sufficient condition for a mapping to be convex. This was originally established in [5] and also proved, essentially as above, in [3]. Actually, the loop of implications is (1) $\implies$ (4) (or $|\varphi| \equiv 1$) $\implies$ (5) $\implies$ (3) $\implies$ (1), and also (1) $\iff$ (2), so all are equivalent to f being a convex mapping.

We now turn to the convexity property of the Poincaré metric. The level set $\lambda(w) = 1/c$ corresponds under f to the curve in $\mathbb{D}$ where

$$(1-|z|^2)|f'(z)| = c\,.$$

This will be a smooth curve provided $\nabla((1-|z|^2)|f'(z)|) \neq 0$ there. Since for real valued functions g one can identify ∇g with the complex number $2\partial_{\bar z} g$, we see that this condition is equivalent to $p \neq 0$ because $\bar p = -(1-|z|^2)\partial_{\bar z}\log(1-|z|^2)|f'(z)|$. This also shows that, thought of as a vector, $\bar p$ is normal to the level sets of the Poincaré metric.

For the proof of Theorem 2 we need a formula for curvature that in itself is not particular to convexity.

Lemma 1. *Let f be locally injective in $\mathbb{D}$, and let $\gamma \subset \mathbb{D}$ be the level set*

$$(1-|z|^2)|f'(z)| = c\,,$$

for a constant c. Suppose that $p \neq 0$ on γ. Then

$$|p(z)|k(z) = 1 + \frac{1}{4}(1-|z|^2)|\frac{f''}{f'}(z)|^2 + \frac{1-|z|^2}{2|p(z)|^2}\mathrm{Re}\{\overline{p(z)}^2 Sf(z)\}\,, \tag{6}$$

where k is the euclidean curvature of γ.

Proof. Because $p \neq 0$ on γ, we may choose a Euclidean arclength parametrization $z = z(s)$, oriented so that the normal direction $\bar p$ points to the right of $z'(s)$. Let $q = \overline{p}$ and $\hat q = q/|q|$. With this orientation of γ

$$z' = i\hat q \quad \text{and} \quad z'' = -k\hat q\,, \tag{7}$$

with $k \geq 0$ if and only if the level set is convex.

Differentiating $(1-|z(s)|^2)|f'(z(s))| = c$ once we obtain

$$\mathrm{Re}\left\{z'\,\frac{f''}{f'}(z)\right\} = 2\frac{\mathrm{Re}\{\bar z z'\}}{1-|z|^2}\,,$$

while a second differentiation yields

$$\begin{aligned}\mathrm{Re}\left\{(z')^2\left(\frac{f''}{f'}\right)'(z)\right\} + \mathrm{Re}\left\{z''\frac{f''}{f'}(z)\right\} &= 2\frac{1+\mathrm{Re}\{\bar z z''\}}{1-|z|^2} + 4\left(\frac{\mathrm{Re}\{\bar z z'\}}{1-|z|^2}\right)^2 \\ &= 2\frac{1+\mathrm{Re}\{\bar z z''\}}{1-|z|^2} + \left(\mathrm{Re}\left\{z'\,\frac{f''}{f'}(z)\right\}\right)^2\,.\end{aligned}$$

Rewrite the last term on the right hand side as

$$\mathrm{Re}\left\{\left(z'\,\frac{f''}{f'}(z)\right)^2\right\}+\left(\mathrm{Im}\left\{z'\,\frac{f''}{f'}(z)\right\}\right)^2$$

to get

$$\mathrm{Re}\left\{(z')^2Sf(z)\right\}+\mathrm{Re}\left\{z''\,\frac{f''}{f'}(z)\right\}=2\frac{1+\mathrm{Re}\{\bar{z}z''\}}{1-|z|^2}+\frac{1}{2}\mathrm{Re}\left\{\left(z'\,\frac{f''}{f'}(z)\right)^2\right\}+\left(\mathrm{Im}\left\{z'\,\frac{f''}{f'}(z)\right\}\right)^2 .$$

Since

$$\frac{1}{2}\mathrm{Re}\left\{\left(z'\,\frac{f''}{f'}(z)\right)^2\right\}=\frac{1}{2}\mathrm{Re}\left\{z'\,\frac{f''}{f'}(z)\right\}^2-\frac{1}{2}\mathrm{Im}\left\{z'\,\frac{f''}{f'}(z)\right\}^2 ,$$

we obtain

$$\mathrm{Re}\left\{(z')^2Sf(z)\right\}+\mathrm{Re}\left\{z''\,\frac{f''}{f'}(z)\right\}=2\frac{1+\mathrm{Re}\{\bar{z}z''\}}{1-|z|^2}+\frac{1}{2}\left|\frac{f''}{f'}(z)\right|^2 ,$$

which we further rewrite as

$$-2\mathrm{Re}\left\{z''p(z)\right\}=\frac{2}{1-|z|^2}+\frac{1}{2}\left|\frac{f''}{f'}(z)\right|^2-\mathrm{Re}\left\{(z')^2Sf(z)\right\} .$$

Using (7), this is the equation in the lemma. □

The issue in establishing strict convexity is the presence of critical points for λ. These correspond to points in $\mathbb{D}$ where $p(z)=0$. Now, convex mappings satisfy

$$\sup_{|z|<1}(1-|z|^2)^2|Sf(z)|\leq 2\,, \tag{8}$$

see [6] and [3]. The results in [2] thus apply, namely that λ has at most one critical point, with the exception of a parallel strip where $\nabla\lambda=0$ all along the central line. For unbounded convex domains, more generally for unbounded domains coming from (8), there are *no* critical points, except again for a parallel strip. When it exists, the unique critical point corresponds to the absolute minimum of λ.

Proof of Theorem 2. We analyze level sets away from the unique critical point of λ, if there is one. Let κ be the curvature of γ relative to the metric $|f'||dz|$ (which is the Euclidean curvature of $f(\gamma)$). Then along γ

$$|f'|\kappa=k-\frac{\partial}{\partial n}\log|f'|\,,$$

where $\partial/\partial n$ is the derivative in the direction $-\hat{q}$. The latter can be computed using

$$\frac{\partial}{\partial n}\log|f'(z)|=\nabla\log|f'(z)|\cdot(-\hat{q}(z))=-2\mathrm{Re}\{\hat{q}(z)\partial_z\log|f'(z)|\},$$

from which

$$
\begin{aligned}
e^{\sigma(z)}\kappa(z) &= k(z) + 2\mathrm{Re}\{\hat{q}(z)\partial_z \log|f'(z)|\} \\
&= k(z) + \frac{\mathrm{Re}\{z\,\frac{f''}{f'}(z)\}}{|p(z)|} - \frac{1}{2|p(z)|}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2 \\
&= \frac{1}{|p(z)|}\left(k|p(z)| + \mathrm{Re}\left\{z\,\frac{f''}{f'}(z)\right\} - \frac{1}{2}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2\right).
\end{aligned}
$$

We replace the expression for $|p|k$ from the lemma, and obtain

$$
e^{\sigma(z)}\kappa(z) = \frac{1}{|p(z)|}\left(1 - \frac{1}{4}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2 + \mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} - \frac{1}{2}(1-|z|^2)\mathrm{Re}\{(z')^2 Sf(z)\}\right)
$$

$$
= \frac{1}{|p(z)|}\left(\mathrm{Re}\left\{1 + z\frac{f''}{f'}(z)\right\} - \frac{1}{4}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2 - \frac{1}{2}(1-|z|^2)\mathrm{Re}\{(z')^2 Sf(z)\}\right)
$$

$$
\geq \frac{1}{|p(z)|}\left(\mathrm{Re}\left\{1 + z\frac{f''}{f'}(z)\right\} - \frac{1}{4}(1-|z|^2)\left|\frac{f''}{f'}(z)\right|^2 - \frac{1}{2}(1-|z|^2)|Sf(z)|\right) \geq 0\,,
$$

the final inequality holding precisely because of Theorem 1.

We claim that if $\kappa = 0$ at one point, then f maps $\mathbb{D}$ onto a half-plane or onto a parallel strip. Indeed, if the curvature vanishes at some point, then all inequalities used to derive that $\kappa \geq 0$ must be equalities. Referring to the proof of Theorem 1, this implies that the function φ must be a constant of absolute value 1 or an automorphism of the disk. In the first case, $f(\mathbb{D})$ is a half-plane, where the level sets of λ are all straight lines parallel to the boundary. In the second case $f(\mathbb{D})$ is a parallel strip. The Poincaré metric on the model strip $|\mathrm{Im}\, y| < \pi/2$, is $\lambda|dw| = \sec y|dw|$, $w = x + iy$. The axis of symmetry $y = 0$ is where λ has its absolute minimum, and $\nabla\lambda = 0$ there. Any other level set will consist of a pair of horizontal lines $y = \pm a$, for some $a \in (0, \pi/2)$. In summary, if the curvature is zero at one point of a level set then it is zero at all points of all level sets.

□

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References

[1] L.A. Caffarelli and A. Friedman, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. 52 (1985), 431-457.

[2] M. Chuaqui and B. Osgood, *Ahlfors-Weill extensions of conformal mappings and critical points of the Poincaré metric*, Comment. Math. Helv. 69 (1994), 659-668.

[3] M. Chuaqui, P. Duren and B. Osgood, *Schwarzian derivatives of convex mappings*, Ann. Acad. Sci. Fenn. Math. 36 (2011), 449-460.

[4] P. Duren, *Univalent Functions*, Springer–Verlag, New York, 1983.

[5] S.-A. Kim and D. Minda, *The hyperbolic and quasihyperbolic metrics in convex regions*, J. Analysis 1 (1993), 109-118.
[6] Z. Nehari, *A property of convex conformal maps*, J. Analyse Math. 30 (1976), 390-393.

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